

Secant varieties to osculating varieties of Veronese embeddings of \mathbb{P}^n .

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ABSTRACT: A well known theorem by Alexander-Hirschowitz states that all the higher secant varieties of $V_{n,d}$ (the d -uple embedding of \mathbb{P}^n) have the expected dimension, with few known exceptions. We study here the same problem for $T_{n,d}$, the tangential variety to $V_{n,d}$, and prove a conjecture, which is the analogous of Alexander-Hirschowitz theorem, for $n \leq 9$. Moreover, we prove that it holds for any n, d if it holds for $d = 3$. Then we generalize to the case of $O_{k,n,d}$, the k -osculating variety to $V_{n,d}$, proving, for $n = 2$, a conjecture that relates the defectivity of $\sigma_s(O_{k,n,d})$ to the Hilbert function of certain sets of fat points in \mathbb{P}^n .

Introduction.

The well known Alexander-Hirschowitz theorem (see [AH1]) states:

Theorem 0.1. (*Alexander-Hirschowitz*) *Let X be a generic collection of s 2-fat points in \mathbb{P}^n_κ . If $(I_X)_d \subset \kappa[x_0, \dots, x_n]$ is the vector space of forms of degree d which are singular at the points of X , then $\dim(I_X)_d = \min\{(n+1)d, \binom{n+d}{n}\}$, as expected, unless:*

- $d = 2, 2 \leq s \leq n$;
- $n = 2, d = 4, s = 5$;
- $n = 3, d = 4, s = 9$;
- $n = 4, d = 3, s = 7$;
- $n = 4, d = 4, s = 14$.

Notice that with “ m -fat point at $P \in \mathbb{P}^n$ ” we mean the scheme defined by the ideal $I_P^m \subset \kappa[x_0, \dots, x_n]$.

An equivalent reformulation of the theorem is in the language of higher secant varieties; let $V_{n,d} \subset \mathbb{P}^N$, with $N = \binom{n+d}{n} - 1$, be the d -ple (Veronese) embedding of \mathbb{P}^n , and let $\sigma_s(V_{n,d})$ be its $(s-1)^{th}$ higher secant variety, that is, the closure of the union of the \mathbb{P}^{s-1} ’s which are s -secant to $V_{n,d}$. Then Theorem 0.1 is equivalent to:

Theorem 0.2. *All the higher secant varieties $\sigma_s(V_{n,d})$ have the expected dimension $\min\{s(n+1)-1, \binom{n+d}{n}-1\}$, unless s, n, d are as in the exceptions of Theorem 0.1.*

An application of the theorem is in terms of the Waring problem for forms (or of the decomposition of a supersymmetric tensor), in fact Theorem 0.1 gives that the general form of degree d in $n+1$ variables can be written as the sum of $\lceil \frac{1}{n+1} \binom{n+d}{d} \rceil$ d th powers of linear forms, with the same list of exceptions (e.g. see [Ge] or [IK]).

In [CGG] a similar problem has been studied, namely whether the dimension of $\sigma_s(T_{n,d})$ is the expected one or not, where $T_{n,d}$ is the tangential variety of the Veronese variety $V_{n,d}$. This too translates into a problem of representation of forms: the generic form parameterized by $\sigma_s(T_{n,d})$ is a form F of degree d which can be written as $F = L_1^{d-1}M_1 + \dots + L_s^{d-1}M_s$, where the L_i, M_i 's are linear forms.

The following conjecture was stated in [CGG]:

Conjecture 1: *The secant variety $\sigma_s(T_{n,d})$ has the expected dimension, $\min\{2sn + s - 1, \binom{n+d}{n} - 1\}$, unless:*

- i) $d = 2$, $2 \leq 2s < n$;
- ii) $d = 3$, $s = n = 2, 3, 4$.

In the same paper the conjecture was proved for $d = 2$ (any s, n) and for $s \leq 5$ (any d, n), while in [B] it is proved for $n = 2, 3$ (any s, d).

In [CGG] (via inverse systems) it is shown that $\sigma_s(T_{n,d})$ is defective if and only if a certain 0-dimensional scheme $Y \subset \mathbb{P}^n$ does not impose independent conditions to forms of degree d in $R := \kappa[x_0, \dots, x_n]$. The scheme $Y = Z_1 \cup \dots \cup Z_s$ is supported at s generic points $P_1, \dots, P_s \in \mathbb{P}^n$, and at each of them the scheme Z_i lies between the 2-fat point and the 3-fat point on P_i (we will call Z_i a $(2, 3, n)$ -scheme, for details see section 1 below).

Hence Conjecture 1 can be reformulated in term of $(I_Y)_d$ having the expected dimension, with the same exceptions, in analogy with the statement of Theorem 0.1.

Theorem 0.1 has been proved thanks to the Horace differential Lemma (AH2, Proposition 9.1; see also here Proposition 1.5) and an induction procedure which has a delicate beginning step for $d = 3$; different proofs for this case are in [Ch1], [Ch2] and in the more recent [BO], where an excellent history of the question can be found.

Also the proof of Conjecture 1 presents the case of $d = 3$ as a crucial one; the first main result in this paper (Corollary 2.5) is to prove that if Conjecture 1 holds for $d = 3$, then it holds also for $d \geq 4$ (and any n, s). The procedure we use is based on Horace differential Lemma too.

We also prove Conjecture 1 for all $n \leq 9$, since with that hypothesis we can check the case $d = 3$ by making use of COCOA (see Corollary 2.4).

A more general problem can be considered (see also [BCGI]): let $O_{k,n,d}$ be the k -osculating variety to $V_{n,d} \subset \mathbb{P}^N$, and study its $(s-1)^{th}$ higher secant variety $\sigma_s(O_{k,n,d})$. Again, we are interested in the problem of determining all s for which $\sigma_s(O_{k,n,d})$ is defective, i.e. for which its dimension is strictly less than its expected dimension (for precise definitions and setting of the problem, see Section 1 of the present paper and in particular Question Q(k,n,d)).

Also in this general case we found in [BCGI] (via inverse systems) that $\sigma_s(O_{k,n,d})$ is defective if and only if a certain 0-dimensional scheme $Y \subset \mathbb{P}^n$ does not impose independent conditions to forms of degree d in $R := \kappa[x_0, \dots, x_n]$. The scheme $Y = Z_1 \cup \dots \cup Z_s$ is supported at s generic points $P_1, \dots, P_s \in \mathbb{P}^n$, and at each of them the ideal of the scheme Z_i is such that $I_{P_i}^{k+2} \subset I_{Z_i} \subset I_{P_i}^{k+1}$ (for details see Lemma 1.2 below).

The following (quite immediate) lemma ([BCGI] 3.1) describes what can be deduced about the postulation of the scheme Y from information on fat points:

Lemma 0.3. *Let P_1, \dots, P_s be generic points in \mathbb{P}^n , and set $X := (k+1)P_1 \cup \dots \cup (k+1)P_s$, $T := (k+2)P_1 \cup \dots \cup (k+2)P_s$. Now let Z_i be a 0-dimensional scheme supported at P_i , $(k+1)P_i \subset Z_i \subset (k+2)P_i$, and set $Y := Z_1 \cup \dots \cup Z_s$. Then, Y is regular in degree d if $h^1(\mathcal{I}_T(d)) = 0$ or if $h^0(\mathcal{I}_X(d)) = 0$.*

Moreover, Y is not regular in degree d if

$$(i) \ h^1(\mathcal{I}_X(d)) > \max\{0, \deg(Y) - \binom{d+n}{n}\},$$

or if

$$(ii) \ h^0(\mathcal{I}_T(d)) > \max\{0, \binom{d+n}{n} - \deg(Y)\}.$$

All cases studied in [BCGI] lead us to state the following:

Conjecture 2a. *The secant variety $\sigma_s(O_{k,n,d})$ is defective if and only if Y is as in case (i) or (ii) of the Lemma above.*

The conjecture amounts to saying that I_Y does not have the expected Hilbert function in degree d only when “forced” by the Hilbert function of one of the fat point schemes X, T .

Notice that (i), respectively (ii), obviously implies that X , respectively T , is defective. Hence, if Conjecture 2a holds and Y is defective in degree d , then either T or X are defective in degree d too, and the defectivity of Y is either given by the defectivity of X or forced by the high defectivity of T .

Thus if the conjecture holds, we have another occurrence of the “ubiquity” of fat points: the problem of $\sigma_s(O_{k,n,d})$ having the right dimension reduces to a problem of computing the Hilbert function in degree d of two schemes of s generic fat points in \mathbb{P}^n , all of them having multiplicity $k+1$, respectively $k+2$.

In [BC] and [BF] the conjecture is proved in \mathbb{P}^2 for $s \leq 9$.

Notice that the Conjecture 2a implies the following one, more geometric, which relates the defectivity of $\sigma_s(O_{k,n,d})$ to the dimensions of the k^{th} and the $(k+1)^{th}$ osculating space at a generic point of the $(s-1)^{th}$ higher secant variety of the Veronese variety $\sigma_s(V_{n,d})$:

Conjecture 2b. *If the secant variety $\sigma_s(O_{k,n,d})$ is defective then at a generic point $P \in \sigma_s(V_{n,d})$, either the k^{th} osculating space $O_{k,\sigma_s(V_{n,d}),P}$ does not have dimension $\min\{s\binom{k+n}{n} - 1, \binom{d+n}{n} - 1\}$, or the $(k+1)^{th}$ osculating space $O_{k+1,\sigma_s(V_{n,d}),P}$ does not have dimension $\min\{s\binom{k+n+1}{n} - 1, \binom{d+n}{n} - 1\}$.*

The implication follows from the fact that (see [BBCF]) for $P \in \langle P_1, \dots, P_s \rangle$:

$$O_{k,\sigma_s(V_{n,d}),P} = \langle O_{k,V_{n,d},P_1}, O_{k,V_{n,d},P_2}, \dots, O_{k,V_{n,d},P_s} \rangle.$$

The other main result in this paper is Theorem 3.5, which proves Conjecture 2a for $n = 2$.

Section 1: Preliminaries and Notations.

In this paper we will always work over a field κ such that $\kappa = \overline{\kappa}$ and $\text{char} \kappa = 0$.

1.1 Notations.

(i) If $P \in \mathbb{P}^n$ is a point and I_P is the ideal of P in \mathbb{P}^n , we denote by mP the fat point of multiplicity m supported at P , i.e. the scheme defined by the ideal I_P^m .

(ii) Let $X \subseteq \mathbb{P}^N$ be a closed irreducible projective variety; the $(s-1)^{th}$ higher secant variety of X is the closure of the union of all linear spaces spanned by s points of X , and it will be denoted by $\sigma_s(X)$.

(iii) Let $X \subset \mathbb{P}^N$ be a variety, and let $P \in X$ be a smooth point; we define the k^{th} osculating space to X at P as the linear space generated by $(k+1)P \cap X$ (i.e. by the k^{th} infinitesimal neighbourhood of P in X) and we denote it by $O_{k,X,P}$; hence $O_{0,X,P} = \{P\}$, and $O_{1,X,P} = T_{X,P}$, the projectivised tangent space to X at P .

Let $U \subset X$ be the dense set of the smooth points where $O_{k,X,P}$ has maximal dimension. The k^{th} osculating variety to X is defined as:

$$O_{k,X} = \overline{\bigcup_{P \in U} O_{k,X,P}}.$$

(iv) We denote by $V_{n,d}$ the d -uple Veronese embedding of \mathbb{P}^n , i.e. the image of the map defined by the linear system of all forms of degree d on \mathbb{P}^n : $\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$.

(v) We denote the k^{th} osculating variety to the Veronese variety by $O_{k,n,d} := O_{k,V_{n,d}}$. When $k = 1$, the osculating variety is called *tangential variety* and it is denoted by $T_{n,d}$.

Hence, the $(s-1)^{th}$ higher secant variety of the k^{th} osculating variety to the Veronese variety $V_{n,d}$ will be denoted by $\sigma_s(O_{k,n,d})$.

Since the case $d \leq k$ is trivial, and the description for $k = 1$ given in [CGG], together with [BCGI, Proposition 4.4] describe the case $d = k + 1$ completely, from now on we make the general assumption, which will be implicit in the rest of the paper, that $d \geq k + 2$.

It is easy to see ([BCGI] 2.3) that the dimension of $O_{k,n,d}$ is always the expected one, that is, $\dim O_{k,n,d} = \min\{N, n + \binom{k+n}{n} - 1\}$. The expected dimension for $\sigma_s(O_{k,n,d})$ is:

$$\expdim \sigma_s(O_{k,n,d}) = \min\{N, s(n + \binom{k+n}{n} - 1) + s - 1\}$$

(there are $\infty^{s(\dim O_{k,n,d})}$ choices of s points on $O_{k,n,d}$, plus ∞^{s-1} choices of a point on the \mathbb{P}^{s-1} spanned by the s points; when this number is too big, we expect that $\sigma_s(O_{k,n,d}) = \mathbb{P}^N$).

When $\dim \sigma_s(O_{k,n,d}) < \expdim \sigma_s(O_{k,n,d})$, the osculating variety is said to be *defective*.

In [BCGI], taking into account that the cases with $n = 1$ can be easily described, while if $n \geq 2$ and $d = k$ one has $\dim \sigma_s(O_{k,n,d}) = N$, we raised the following question:

Question Q(k,n,d): For all k, n, d such that $d \geq k + 1$, $n \geq 2$, describe all s for which $\sigma_s(O_{k,n,d})$ is defective, i.e.

$$\dim \sigma_s(O_{k,n,d}) < \min\{N, s(n + \binom{k+n}{n} - 1) + s - 1\} = \min\left\{\binom{d+n}{n} - 1, s\binom{k+n}{n} + sn - 1\right\}.$$

We were able to answer the question for s, n, d, k in several ranges, thanks to the following lemma (see [BCGI] 2.11 and results of Section 2):

Lemma 1.2 For any $k, n, d \in \mathbb{N}$ such that $n \geq 2$, $d \geq k + 1$, there exists a 0-dimensional subscheme $Z = Z(k, n) \in \mathbb{P}^n$ depending only from k and n and not from d , such that:

(a) Z is supported on a point P , and one has:

$$(k+1)P \subset Z(k, n) \subset (k+2)P, \quad \text{with} \quad l(Z) = \binom{k+n}{n} + n;$$

(b) denoting by $Y = Y(k, n, s)$ the generic union in \mathbb{P}^n of Z_1, \dots, Z_s where $Z_i \cong Z$ for $i = 1, \dots, s$, then

$$\dim \sigma_s(O_{k,n,d}) = \expdim \sigma_s(O_{k,n,d}) - h^0(\mathcal{I}_Y(d)) + \max\{0, \binom{d+n}{n} - l(Y)\}$$

In particular, $\sigma_s(O_{k,n,d})$ is not defective if and only if Y is regular in degree d , i.e. $h^0(\mathcal{I}_Y(d)) \cdot h^1(\mathcal{I}_Y(d)) = 0$.

The homogeneous ideal of this 0-dimensional scheme Z is defined in [BCGI] 2.5 through inverse systems, so we don't have an explicit geometric description of it in the general case. Anyway, for $k = 1$ it is possible to describe it geometrically as follows (see [CGG] Section 2):

Definition 1.3. Let P be a point in \mathbb{P}^n , and L a line through P ; we say that a 0-dimensional scheme $X \subset \mathbb{P}^n$ is a $(2, 3, n)$ -scheme supported on P with direction L if $I_X = I_P^3 + I_L^2$. Hence, the length of a $(2, 3, n)$ -point is $2n + 1$. The scheme $Z(1, n)$ of Lemma 1.2 is a $(2, 3, n)$ -scheme.

We say that a subscheme of \mathbb{P}^n is a generic union of s $(2, 3, n)$ -schemes if it is the union of X_1, \dots, X_s where X_i is a $(2, 3, n)$ -scheme supported on P_i with direction L_i , with P_1, \dots, P_s generic points and L_1, \dots, L_s generic lines through P_1, \dots, P_s .

We are going to use these schemes in Section 2, so we need to know more about them; but first we recall the Differential Horace Lemma of [AH2], writing it in the context where we shall use it.

Definition 1.4. In the algebra of formal functions $\kappa[[\mathbf{x}, y]]$, where $\mathbf{x} = (x_1, \dots, x_{n-1})$, a *vertically graded* (with respect to y) ideal is an ideal of the form:

$$I = I_0 \oplus I_1 y \oplus \dots \oplus I_{m-1} y^{m-1} \oplus (y^m)$$

where for $i = 0, \dots, m-1$, $I_i \subset \kappa[[\mathbf{x}]]$ is an ideal.

Let Q be a smooth n -dimensional integral scheme, let K be a smooth irreducible divisor on Q . We say that $Z \subset Q$ is a *vertically graded subscheme* of Q with base K and support $z \in K$, if Z is a 0-dimensional scheme with support at the point z such that there is a regular system of parameters (\mathbf{x}, y) at z such that $y = 0$ is a local equation for K and the ideal of Z in $\widehat{\mathcal{O}}_{Q,z} \cong \kappa[[\mathbf{x}, y]]$ is vertically graded.

Let $Z \subset Q$ be a vertically graded subscheme with base K , and $p \geq 0$ be a fixed integer; we denote by $Res_K^p(Z) \subset Q$ and $Tr_K^p(Z) \subset K$ the closed subschemes defined, respectively, by the ideals:

$$\mathcal{I}_{Res_K^p(Z)} := \mathcal{I}_Z + (\mathcal{I}_Z : \mathcal{I}_K^{p+1}) \mathcal{I}_K^p, \quad \mathcal{I}_{Tr_K^p(Z), K} := (\mathcal{I}_Z : \mathcal{I}_K^p) \otimes \mathcal{O}_K.$$

In $Res_K^p(Z)$ we take away from Z the $(p+1)^{th}$ "slice"; in $Tr_K^p(Z)$ we consider only the $(p+1)^{th}$ "slice". Notice that for $p = 0$ we get the usual trace and residual schemes: $Tr_K(Z)$ and $Res_K(Z)$.

Finally, let $Z_1, \dots, Z_r \subset Q$ be vertically graded subschemes with base K and support z_i , $Z = Z_1 \cup \dots \cup Z_r$, and $\mathbf{p} = (p_1, \dots, p_r) \in \mathbb{N}^r$.

We set:

$$Tr_K^{\mathbf{p}}(Z) := Tr_K^{p_1}(Z_1) \cup \dots \cup Tr_K^{p_r}(Z_r), \quad Res_K^{\mathbf{p}}(Z) := Res_K^{p_1}(Z_1) \cup \dots \cup Res_K^{p_r}(Z_r).$$

Proposition 1.5. (Horace differential Lemma, [AH2] Proposition 9.1) *Let H be a hyperplane in \mathbb{P}^n and let $W \subset \mathbb{P}^n$ be a 0-dimensional closed subscheme.*

Let $S_1, \dots, S_r, Z_1, \dots, Z_r$ be 0-dimensional irreducible subschemes of \mathbb{P}^n such that $S_i \cong Z_i$, $i = 1, \dots, r$, Z_i has support on H and is vertically graded with base H , and the supports of $S = S_1 \cup \dots \cup S_r$ and $Z = Z_1 \cup \dots \cup Z_r$ are generic in their respective Hilbert schemes. Let $\mathbf{p} = (p_1, \dots, p_r) \in \mathbb{N}^r$. Assume:

- a) $H^0(\mathcal{I}_{Tr_H W \cup Tr_H^{\mathbf{p}}(Z), H}(n)) = 0$ and
- b) $H^0(\mathcal{I}_{Res_H W \cup Res_H^{\mathbf{p}}(Z)}(n-1)) = 0$,

then

$$H^0(\mathcal{I}_{W \cup S}(n)) = 0.$$

Definition 1.6. A 2-jet is a 0-dimensional scheme $J \subset \mathbb{P}^n$ with support at a point $P \in \mathbb{P}^n$ and degree 2; namely the ideal of J is of type: $I_P^2 + I_L$, where $L \subset \mathbb{P}^n$ is a line containing P . We will say that J_1, \dots, J_s are generic in \mathbb{P}^n , if the points P_1, \dots, P_s are generic in \mathbb{P}^n and L_1, \dots, L_s are generic lines through P_1, \dots, P_s .

Remark 1.7. Let $X \subset \mathbb{P}^n$ be a $(2, 3, n)$ -scheme supported at P with direction L and (y_1, \dots, y_n) be local coordinates around P , such that L becomes the y_n -axis; then, $I_X = (y_1 y_n^2, \dots, y_{n-1} y_n^2, y_n^3, y_1^2, y_1 y_2, \dots, y_{n-1}^2)$ (y_n appears only in the first n generators). Let H , respectively K , be a hyperplane through L , respectively transversal to L ; then, we can assume $I_H = (y_{n-1})$, respectively $I_K = (y_n)$. We now compute $Res_H^{\mathbf{p}}(X)$ and $Tr_H^{\mathbf{p}}(X)$. One has:

- a) $Res_H X = Res_H^0(X)$, $I_{Res_H(X)} = (I_X : y_{n-1}) = (y_1, \dots, y_{n-1}, y_n^2)$, hence $Res_H X$ is a 2-jet lying on L ;
- b) $Tr_H(X) = Tr_H^0(X)$, $I_{Tr_H(X)} = I_X + (y_{n-1}) = (y_1 y_n^2, \dots, y_{n-2} y_n^2, y_n^3, y_1^2, y_1 y_2, \dots, y_{n-2}^2)$, hence $Tr_H(X)$ is a $(2, 3, n-1)$ -scheme of H .

Hence the scheme X as a vertically graded scheme with base H has only two layers (strata); in other words, $Tr_H^{\mathbf{p}}(X)$ is empty for $p > 1$, and $Res_H^1(X)$ is a $(2, 3, n-1)$ -scheme of H , while $Tr_H^1(X)$ is a 2-jet lying on L .

Now we want to compute $Res_K^{\mathbf{p}}(X)$ and $Tr_K^{\mathbf{p}}(X)$. Consider first:

- b) $I_{Tr_K(X)} = I_X + (y_n) = (y_n, y_1^2, y_1 y_2, \dots, y_{n-1}^2)$, hence $Tr_K(X)$ is a 2-fat point of $K \cong \mathbb{P}^{n-1}$,
- a) $I_{Res_K X} = (I_X : y_n) = (y_1 y_n, \dots, y_{n-1} y_n, y_n^2, y_1^2, y_1 y_2, \dots, y_{n-1}^2)$, hence $Res_K X$ is a 2-fat point of \mathbb{P}^n .

So the scheme X , as a vertically graded scheme with base K , has only three layers (strata); the 0-layer is $Tr_K(X) = Tr_K^0(X)$, the 1-layer is the 0-layer of $Res_K X = Res_K^0(X)$, hence it is again a 2-fat point of $K \cong \mathbb{P}^{n-1}$, and the 2-layer is the 1-layer of $Res_K X$, hence it is a point of \mathbb{P}^n . In other words, $Tr_H^{\mathbf{p}}(X)$ is empty for $p > 2$, $Res_K^1(X)$ is a 2-fat point of \mathbb{P}^n , while $Tr_K^1(X)$ is a 2-fat point of K ; $Res_K^2(X)$ is a 2-fat point of K doubled in a direction transversal to K (i.e., $I_{Res_K^2(X)} = (y_n^2, y_1^2, y_1 y_2, \dots, y_{n-1}^2)$), while $Tr_K^2(X)$ is a point of \mathbb{P}^n .

We will use in the sequel the fact that by adding s generic 2-jets to any 0-dimensional scheme $Z \subset \mathbb{P}^n$ we impose a maximal number of independent conditions to forms in $I_Z(d)$, for all d . This is probably classically known, but we write a proof here for lack of a reference:

Lemma 1.8 *Let $Z \subseteq \mathbb{P}^n$ be a scheme, and let $J \subset \mathbb{P}^n$ be a generic 2-jet. Then:*

$$h^0(\mathcal{I}_{Z \cup J}(d)) = \max\{h^0(\mathcal{I}_Z(d)) - 2, 0\}.$$

Proof: Let P be the support of J ; then we know that $h^0(\mathcal{I}_{Z \cup P}(d)) = \max\{h^0(\mathcal{I}_Z(d)) - 1, 0\}$, so if $h^0(\mathcal{I}_Z(d)) \leq 1$ there is nothing to prove. Let $h^0(\mathcal{I}_Z(d)) \geq 2$, then $h^0(\mathcal{I}_{Z \cup P}(d)) = h^0(\mathcal{I}_Z(d)) - 1 \geq 1$. Since J is generic, if $h^0(\mathcal{I}_{Z \cup J}(d)) = h^0(\mathcal{I}_{Z \cup P}(d))$, then every form of degree d containing $Z \cup P$ should have double intersection with almost every line containing P , hence it should be singular at P . This means that when we force a form in the linear system $|H^0(\mathcal{I}_Z(d))|$ to vanish at P , then we are automatically imposing to the form to be singular at P , and this holds for P in a dense open set of \mathbb{P}^n , say U . If the form f is generic in $|H^0(\mathcal{I}_Z(d))|$, its zero set V meets U in a non empty subset of V , so f is singular at whatever point P' we choose in $V \cap U$, and this means that the hypersurface V is not reduced. Since the dimension of the linear system $|H^0(\mathcal{I}_Z(d))|$ is at least 2, this is impossible by Bertini Theorem (e.g. see [J], Theorem 6.3). \square

Let $Z \subseteq \mathbb{P}^n$ be a zero-dimensional scheme; the following simple Lemma gives a criterion for adding to Z a scheme D which lies on a smooth hypersurface $\mathcal{F} \subseteq \mathbb{P}^n$ and is made of s generic 2-jets on \mathcal{F} , in such a way that D imposes independent conditions to forms of a given degree in the ideal of Z (see Lemma 4 in [Ch1] and Lemma 1.9 in [CGG2] for the case of simple points on a hypersurface).

Lemma 1.9 *Let $Z \subseteq \mathbb{P}^n$ be a zero dimensional scheme. Let $\mathcal{F} \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d and let $Z' = \text{Res}_{\mathcal{F}} Z$. Let P_1, \dots, P_s be generic points on \mathcal{F} , let L_1, \dots, L_s lines with $P_i \in L_i$, and such that each line L_i is generic in $T_{P_i}(\mathcal{F})$; let J_i be the 2-jet with support at P_i and contained in L_i . We denote by $D_s = J_1 \cup \dots \cup J_s$ the union of these s 2-jets generic in \mathcal{F} .*

- i) If $\dim(I_{Z+D_{s-1}})_t \geq \dim(I_{Z'})_{t-d} + 2$, then $\dim(I_{Z+D_s})_t = \dim(I_Z)_t - 2s$;*
- ii) if $\dim(I_{Z'})_{t-d} = 0$ and $\dim(I_Z)_t \leq 2s$, then $\dim(I_{Z+D_s})_t = 0$.*

Proof: i) By induction on s . If $s = 1$, by assumption $\dim(I_Z)_t \geq \dim(I_{Z'})_{t-d} + 2$, hence in the exact sequence $0 \rightarrow H^0(\mathcal{I}_{Z'}(t-d)) \xrightarrow{\phi} H^0(\mathcal{I}_Z(t-d)) \rightarrow H^0(\mathcal{I}_{Z \cap \mathcal{F}, \mathcal{F}}(t)) \rightarrow \dots$ the cokernel of the map ϕ has dimension at least 2 and so $(I_Z)_t$ cuts on \mathcal{F} a linear system (i.e. $|H^0(\mathcal{I}_{Z \cap \mathcal{F}, \mathcal{F}}(t))|$) of (projective) dimension ≥ 1 . We have $\dim(I_{Z+P_1})_t = \dim(I_Z)_t - 1$, since otherwise each hypersurface in $|(I_Z)_t|$ would contain the generic point P_1 of \mathcal{F} , that is, would contain \mathcal{F} .

Assume $\dim(I_{Z+J_1})_t = \dim(I_{Z+P_1})_t = \dim(I_Z)_t - 1$; this means that if we impose to $S \in |(I_Z)_t|$ the passage through P_1 automatically we impose to S to be tangent to L_1 at P_1 , and L_1 being generic in $T_{P_1}(\mathcal{F})$, this means that each S passing through P_1 is tangent to \mathcal{F} at P_1 . Let's say that this holds for P_1 in the open not empty subset U of \mathcal{F} ; for S generic in $|(I_Z)_t|$, $U' = S \cap \mathcal{F} \cap U$ is not empty, hence the generic S is tangent to \mathcal{F} at each $P \in U'$. This means that $|(I_Z)_t|$ cuts on \mathcal{F} a linear system of positive dimension whose generic element is generically non reduced, and this is impossible, by Bertini Theorem (e.g. see [J], Theorem 6.3).

Now let $s > 1$. Since $\dim(I_{Z+D_{s-2}})_t \geq \dim(I_{Z+D_{s-1}})_t > \dim(I_{Z'})_{t-d}$ by assumption, and $\text{Res}_{\mathcal{F}}(Z + D_{s-1}) = Z'$, the case $s = 1$ gives $\dim(I_{Z+D_s})_t = \dim(I_{Z+D_{s-1}})_t - 2$. So, by the induction hypothesis, we get

$$\dim(I_{Z+D_s})_t = (\dim(I_Z)_t - 2(s-1)) - 2 = \dim(I_Z)_t - 2s.$$

ii) Assume first $\dim(I_Z)_t \leq 2$; it is enough to prove $\dim(I_{Z+J_1})_t = 0$ since then also $\dim(I_{Z+D_s})_t = 0$. If $\dim(I_Z)_t = 2$ this follows by *i*) and if $\dim(I_Z)_t = 0$ this is trivial. If $\dim(I_Z)_t = 1$, then if $\dim(I_{Z+P_1})_t = 0$ we are done. If $\dim(I_{Z+P_1})_t = 1$, then by the genericity of P_1 we have that the unique S in the system contains \mathcal{F} , i.e. $S = \mathcal{F} \cup G$, but then $Z' \subseteq G$, which contradicts $\dim(I_{Z'})_{t-d} = 0$.

Otherwise, let $\dim(I_Z)_t = 2v + \delta \geq 3$, $\delta = 0, 1$. If $\delta = 0$, then $\dim(I_{Z+D_{v-1}})_t \geq 2 = \dim(I_{Z'})_{t-d} + 2$, and by *i*) we get $\dim(I_{Z+D_v})_t = \dim(I_Z)_t - 2v = 0$, and, since $s \geq v$, it follows that $\dim(I_{Z+D_s})_t = 0$.

If $\delta = 1$, then $\dim(I_{Z+D_{v-1}})_t \geq 3 \geq \dim(I_{Z'})_{t-d} + 2$, and, by *i*), $\dim(I_{Z+D_{v-1}})_t = 3$ and $\dim(I_{Z+D_v})_t = \dim(I_Z)_t - 2v = 1$. Notice that the only element in $(I_{Z+D_v})_t$ cannot have \mathcal{F} as a fixed component, otherwise we would have $\dim(I_{Z'})_{t-d} = 1$ and not $= 0$; hence $\dim(I_{Z+D_v+P_{v+1}})_t = 0$ and so, since $2s \geq 2v + 1$ and $D_v \cup P_{v+1} \subset D_s$, $\dim(I_{D_s})_t = 0$.

□

Now we give a Lemma which will be of use in the proof of Theorem 2.2.

Lemma 1.10 *Let $R \subseteq \mathbb{P}^n$ be a zero dimensional scheme contained in a $(2, 3, n)$ -scheme with $r = \deg Y \leq 2n$; assume moreover that, if $r \geq n + 1$, then R is a flat limit of the union of a 2-fat point of \mathbb{P}^n and of a scheme (eventually empty) contained in a 2-fat point of a \mathbb{P}^{n-1} , and that, if $r \leq n$, then R is contained in a 2-fat point of a \mathbb{P}^{n-1} . Then, there exists a flat family for which R is a special fiber and the generic fiber is the generic union in \mathbb{P}^n of δ 2-fat points, h 2-jets and ϵ simple points, where $r = (n + 1)\delta + 2h + \epsilon$, $0 \leq \delta \leq 1$, $0 \leq \epsilon \leq 1$, and $2h + \epsilon \leq n$.*

Proof: In the following we denote by $2_t P$ a 2-fat point of a linear variety $K \subseteq \mathbb{P}^n$, $K \cong \mathbb{P}^t$. We first notice that if A is a subscheme of $2_n P$ with $\deg A = n$ then A is a scheme of type $2_{n-1} P$. The proof is by induction on n : if $n = 2$, the statement is trivial since the only scheme of degree 2 in \mathbb{P}^2 is a 2-jet, i.e. a $2_1 P$. Now assume the assertion true for $n - 1$, let A be a subscheme of $2_n P$ with $\deg A = n$ and let H be a hyperplane through the support of A . Since $\deg 2_n P \cap H = n$, we have $n - 1 \leq \deg A \cap H \leq n$. If $\deg A \cap H = n$ then $A = 2_{n-1} P$ and we are done. If $\deg A \cap H = n - 1$ then $\text{Res}_H A$ is a simple point, and by induction $A \cap H = 2_{n-2} P$. Hence there is a hyperplane K such that $A \cap H$ is a 2-fat point of $H \cap K$, and working for example in affine coordinates, it is easy to see that A is a 2-fat point of the \mathbb{P}^{n-1} generated by $H \cap K$ and a normal direction to H .

In order to prove the Lemma, it is enough to prove that the generic union in \mathbb{P}^n of h 2-jets and ϵ simple points, with $0 \leq \epsilon \leq 1$ and $2h + \epsilon \leq n$, specializes to any possible subscheme M of a scheme of type $2_{n-1} P$: in fact, if $r \leq n$ we are done, if $r \geq n + 1$, the collision of a $2_n P$ with M gives R .

By induction on n : if $n = 2$, the statement is trivial. Let us now consider the generic union of h 2-jets and ϵ simple points in \mathbb{P}^n , with $0 \leq \epsilon \leq 1$ and $2h + \epsilon \leq n$. We have two cases.

Case 1: if $2h + \epsilon \leq n - 1$, we specialize everything inside a hyperplane H where, by induction assumption, this scheme specializes to any possible subscheme of a scheme of type $2_{n-2} P$, i.e., to any possible subscheme of degree $\leq n - 1$ of a scheme of type $2_{n-1} P$.

Case 2: If $2h + \epsilon = n$, we have to show that the generic union of h 2-jets and ϵ simple points specializes to a scheme $2_{n-1} P$.

If n is odd, then $h = \frac{n-1}{2}$ and $\epsilon = 1$; by induction assumption, $\frac{n-1}{2}$ 2-jets specialize to a scheme of type $2_{n-2} P$, and the generic union of the last one with a simple point specializes to a scheme of type $2_{n-1} P$.

If n is even, then $h = \frac{n}{2}$ and $\epsilon = 0$; by induction assumption, $\frac{n}{2} - 1$ 2-jets specialize to a scheme of degree $n - 2$ contained in a scheme of type $2_{n-2}P$, which is a $2_{n-3}P$, so it is enough to prove that the generic union of the last one with a 2-jet specializes to a scheme of type $2_{n-1}P$.

In affine coordinates x_1, \dots, x_n , let $x_{n-2} = x_{n-1} = x_n = 0$ be the linear subspace containing $2_{n-3}P$, so that $I_{2_{n-3}P} = (x_1, \dots, x_{n-3})^2 \cap (x_{n-2}, x_{n-1}, x_n)$, and let $(x_1, \dots, x_{n-3}, x_{n-2} - a, x_{n-1}^2, x_n)$ be the ideal of a 2-jet moving along the x_{n-2} -axis; then it is immediate to see that the limit for $a \rightarrow 0$ of $(x_1, \dots, x_{n-3})^2 \cap (x_{n-2}, x_{n-1}, x_n) \cap (x_1, \dots, x_{n-3}, x_{n-2} - a, x_{n-1}^2, x_n)$ is $(x_1, \dots, x_{n-1})^2 \cap (x_n)$, which is the ideal of a $2_{n-1}P$. \square

2. On Conjecture 1.

We want to study $\sigma_s(T_{n,d})$, and we have seen that its dimension is given by the Hilbert function of s generic $(2, 3, n)$ -points in \mathbb{P}^n .

Definition 2.0 For each n and d we define $s_{n,d}, r_{n,d} \in \mathbb{N}$ as the two positive integers such that

$$\binom{d+n}{n} = (2n+1)s_{n,d} + r_{n,d}, \quad 0 \leq r_{n,d} < 2n+1.$$

In the following we denote by $X_{s,n} \subset \mathbb{P}^n$ the zero dimensional scheme union of s generic $(2, 3, n)$ -schemes A_1, \dots, A_s . We also denote by $X_{s_{n,d}}$ the scheme $X_{s,n}$, with $s = s_{n,d}$. Hence $X_{s_{n,d}}$ is the union of the maximum number of generic $(2, 3, n)$ -points that we expect to impose independent conditions to forms of degree d . We will also use $X_{s_{n,d}+1}$ to indicate $X_{s+1,n}$ when $s = s_{n,d}$.

With $Y_{n,d} \subset \mathbb{P}^n$ we denote a scheme generic union of $X_{s_{n,d}}$ and $R_{n,d}$, where $R_{n,d}$ is a zero dimensional scheme contained in a $(2, 3, n)$ -point, with $\deg(R_{n,d}) = r_{n,d}$.

A 0-dimensional subscheme A of \mathbb{P}^n is said to be “ $\mathcal{O}_{\mathbb{P}^n}(d)$ -numerically settled” if $\deg A = h^0(\mathcal{O}_{\mathbb{P}^n}(d))$; in this case, $h^0(\mathcal{I}_A(d)) = 0$ if and only if $h^1(\mathcal{I}_A(d)) = 0$. The scheme $Y_{n,d}$ is $\mathcal{O}_{\mathbb{P}^n}(d)$ -numerically settled for all n, d .

Remark 2.1 Let A be a 0-dimensional $\mathcal{O}_{\mathbb{P}^n}(d)$ -numerically settled subscheme of \mathbb{P}^n , and assume $h^0(\mathcal{I}_A(d)) = 0$. Let $B \subseteq A$ and $C \supseteq A$ be 0-dimensional subschemes of \mathbb{P}^n ; then, $h^0(\mathcal{I}_C(d)) = 0$, and $h^1(\mathcal{I}_B(d)) = 0$, or equivalently, $h^0(\mathcal{I}_B(d)) = \deg A - \deg B$.

Hence if we prove $h^0(\mathcal{I}_{Y_{n,d}}(d)) = 0$ then we know that $h^1(\mathcal{I}_{Y_{n,d}}(d)) = 0$, and

$$h^0(\mathcal{I}_{X_{s,n}}(d)) = 0 \text{ for all } s > s_{n,d},$$

$$h^1(\mathcal{I}_{X_{s,n}}(d)) = 0 \text{ for all } s \leq s_{n,d}.$$

Moreover, if $h^0(\mathcal{I}_{Y_{n,d}}(d)) = 0$ then also $h^0(\mathcal{I}_D(d)) = 0$, where D denotes a generic union of $X_{s_{n,d}}$, of $\lfloor \frac{r_{n,d}}{2} \rfloor$ 2-jets and of $r_{n,d} - 2\lfloor \frac{r_{n,d}}{2} \rfloor$ simple points. In fact, we have $h^0(\mathcal{I}_{X_{s_{n,d}}}(d)) = \deg(R_{n,d}) = r_{n,d}$ and we conclude by Lemma 1.8.

The same conclusion (i.e. $h^0(\mathcal{I}_D(d)) = 0$) holds in the weaker assumption that $h^1(\mathcal{I}_{X_{s_{n,d}}}(d)) = 0$, since in this case $h^0(\mathcal{I}_{X_{s_{n,d}}}(d)) = \binom{d+n}{n} - \deg(X_{s_{n,d}}) = r_{n,d}$ and we get $h^0(\mathcal{I}_D(d)) = 0$ by Lemma 1.8.

Theorem 2.2 Suppose that for all $n \geq 5$, we have $h^1(\mathcal{I}_{X_{s_{n,3}}}(3)) = 0$ and $h^0(\mathcal{I}_{X_{s_{n,3}+1}}(3)) = 0$; then $h^0(\mathcal{I}_{Y_{n,d}}(d)) = h^1(\mathcal{I}_{Y_{n,d}}(d)) = 0$, for all $d \geq 4$, $n \geq 4$.

Proof: Let us consider a hyperplane $H \subset \mathbb{P}^n$; we want a scheme Z with support on H , made of $(2, 3, n)$ -schemes, and an integer vector \mathbf{p} , such that the “differential trace” $Tr_H^{\mathbf{p}}(Z) \subset H$ is $\mathcal{O}_{\mathbb{P}^{n-1}}(d)$ -numerically settled.

Let us consider $n \geq 5$ first. Since $0 \leq r_{n-1,d} \leq 2n-2$, we write $r_{n-1,d} = n\delta + 2h + \epsilon$, with $0 \leq \epsilon \leq 1$, $0 \leq \delta \leq 1$ and $2h + \epsilon \leq n$.

We denote by Z the zero dimensional scheme union of $s_{n-1,d} + h + \epsilon + \delta$ (hence $\delta = 0$ if $0 \leq r_{n-1,d} \leq n$, while $\delta = 1$ if $n+1 \leq r_{n-1,d} \leq 2n-2$), $(2, 3, n)$ -schemes $Z_1, \dots, Z_{s_{n-1,d}+h+\epsilon+\delta}$, where each Z_i is supported at P_i with direction L_i , and:

- the P_i 's are generic on H , $i = 1, \dots, s_{n-1,d} + h + \epsilon + \delta$;
- $L_i \subset H$ for $i = 1, \dots, s_{n-1,d} + h$;
- if $(\epsilon, \delta) \neq (0, 0)$, the corresponding lines $L_{s_{n-1,d}+h+1}, L_{s_{n-1,d}+h+2}$ have generic directions in \mathbb{P}^n (hence not contained in H).

In case $n = 4$, instead, we write $r_{3,d} = 2h + \epsilon$, with $0 \leq \epsilon \leq 1$, and Z is given as before. Notice that in this case $0 \leq h \leq 3$, and it can appear only one line $L_{s_{3,d}+h+1}$, not contained in H .

We want to use the Horace differential Lemma 1.5, where the role of the schemes H and Z appearing in the statement of the Lemma are played by our hyperplane H and the scheme Z just defined, and with:

$$W = A_{s_{n-1,d}+h+\epsilon+1} \cup \dots \cup A_{s_{n,d}} \cup R_{n,d},$$

$$S = A_1 \cup \dots \cup A_{s_{n-1,d}+h+\epsilon+\delta},$$

$$\mathbf{p} = (\underbrace{0, \dots, 0}_{s_{n-1,d}}, \underbrace{1, \dots, 1}_h, \underbrace{2}_{\epsilon}, \underbrace{0}_{\delta}).$$

so that $Tr_H W = \emptyset$ and $Res_H W = W$, and $Y_{n,d} = W \cup S$.

Notice that this construction is possible, since $s_{n-1,d} + h + 2 \leq s_{n,d}$ (and even more than that): see Appendix A, A.1.

In order to simplify notations, we set:

$$T_i^j := Tr_H^j(Z_i), \quad R_i^j := Res_H^j(Z_i), \quad j = 0, 1, 2, \quad i = 1, \dots, s_{n-1,d} + h + \epsilon + \delta,$$

$$T := Tr_H W \cup Tr_H^{\mathbf{p}}(Z) = T_1^0 \cup \dots \cup T_{s_{n-1,d}}^0 \cup T_{s_{n-1,d}+1}^1 \cup \dots \cup T_{s_{n-1,d}+h}^1 \cup T_{s_{n-1,d}+h+\epsilon}^2 \cup T_{s_{n-1,d}+h+\epsilon+\delta}^0,$$

$$R := Res_H W \cup Res_H^{\mathbf{p}}(Z) = W \cup R_1^0 \cup \dots \cup R_{s_{n-1,d}}^0 \cup R_{s_{n-1,d}+1}^1 \cup \dots \cup R_{s_{n-1,d}+h}^1 \cup R_{s_{n-1,d}+h+\epsilon}^2 \cup R_{s_{n-1,d}+h+\epsilon+\delta}^0.$$

Observe that, by Remark 1.7 :

$T_1^0, \dots, T_{s_{n-1,d}}^0$ are $(2, 3, n-1)$ -points in $H \cong \mathbb{P}^{n-1}$, and $R_1^0, \dots, R_{s_{n-1,d}}^0$ are 2-jets in H ;

$T_{s_{n-1,d}+1}^1, \dots, T_{s_{n-1,d}+h}^1$ are 2-jets in H and $R_{s_{n-1,d}+1}^1, \dots, R_{s_{n-1,d}+h}^1$ are $(2, 3, n-1)$ -points in H ;

$T_{s_{n-1,d}+h+\epsilon}^2$ is, when appearing, a simple point of H , and $R_{s_{n-1,d}+h+\epsilon+\delta}^2$ is a 2-fat point of H doubled in a direction transversal to H ;

$T_{s_{n-1,d}+h+\epsilon+\delta}^0$ is, when appearing, a 2-fat point on H , and $R_{s_{n-1,d}+h+\epsilon}^0$ is a 2-fat point in \mathbb{P}^n with support on H .

We will also make use of the scheme:

$$B := W \cup R_{s_{n-1,d}+1}^1 \cup \dots \cup R_{s_{n-1,d}+h}^1 \cup R_{s_{n-1,d}+h+\epsilon}^2.$$

Let us consider the following four statements:

$$\mathbf{Prop}(n, d) : h^0(\mathcal{I}_{Y_{n,d}}(d)) = 0; \quad \mathbf{Reg}(n, d) : h^1(\mathcal{I}_{X_{s,n}}(d)) = 0 \text{ and } h^0(\mathcal{I}_{X_{s,n}+1}(d)) = 0,$$

$$\mathbf{Degue}(n, d) : h^0(\mathcal{I}_R(d-1)) = 0; \quad \mathbf{Dime}(n, d) : h^0(\mathcal{I}_{T,H}(d)) = 0.$$

If $\mathbf{Degue}(n, d)$ and $\mathbf{Dime}(n, d)$ are true, we know that $\mathbf{Prop}(n, d)$ is true too, by Proposition 1.5.

For the first values of n, d , we will need an “ad hoc” construction, which is given by the following:

Lemma 2.3 *Let $d = 4$ and $n \in \{4, 5, 6\}$, then $\mathbf{Prop}(n, d)$ holds.*

Proof of the Lemma.

Case $n = 4$. Here we use the construction of R and T described above, hence we need to show that $\mathbf{Degue}(4, 4)$ and $\mathbf{Dime}(4, 4)$ hold. Since $s_{3,4} = 5$, and $r_{3,4} = 0$, T is made of five generic $(2, 3, 3)$ -points in $H \cong \mathbb{P}^3$, so $\mathbf{Dime}(4, 4)$ holds (i.e. $h^0(\mathbb{P}^3, \mathcal{I}_{T,H}(4)) = h^0(\mathbb{P}^3, \mathcal{I}_{X_{5,3}}(4)) = 0$), e.g. see [CGG1].

In order to prove $\mathbf{Degue}(4, 4)$ we want to apply Lemma 1.2, with R made of five 2-jets plus the scheme $B = W$; hence we need to show that $h^0(\mathcal{I}_B(3)) \leq 10$, while $h^0(\mathcal{I}_{Res_H(B)}(2)) = 0$. Since here $s_{4,4} = 7 = r_{4,4}$, while $r_{3,4} = 0$, we have that $B = W = Res_H(B)$ and it is given by A_6 and A_7 , plus $R_{4,4}$. Hence we have $h^1(\mathcal{I}_B(3)) = 0$, since B is contained in the scheme made of 3 generic $(2, 3, 4)$ -points (which is known to have maximal Hilbert function, by [CGG1] or [B]); $h^1(\mathcal{I}_B(3)) = 0$ is equivalent to saying that $h^0(\mathcal{I}_B(3)) = 2s_{3,4} = 10$, as required. Moreover $h^0(\mathcal{I}_B(2)) = 0$, since there is one only form of degree two passing through two generic $(2, 3, 4)$ -points in \mathbb{P}^4 , given by the hyperplane containing the two double lines, doubled. Since the support of $R_{4,4}$ is generic, we get $h^0(\mathcal{I}_B(2)) = 0$. So we have that $\mathbf{Degue}(4, 4)$ holds, and $\mathbf{Prop}(4, 4)$ holds too.

Case $n = 5$. Here we need to use a different construction. We have $s_{5,4} = 11$, $r_{5,4} = 5$, $s_{4,4} = 7 = r_{4,4}$. We want to use the Horace differential Lemma 1.5 with $Z = Z_1 \cup \dots \cup Z_8 \cup R_{5,4}$, where Z_1, \dots, Z_8 are $(2, 3, 5)$ schemes supported at generic points of H with direction $L_1, \dots, L_8 \subset H$, and we specialize $R_{5,4}$ so that $R_{5,4} \subset H$, contained in a generic $(2, 3, 4)$ -scheme of H ; with $W = A_9 \cup A_{10} \cup A_{11}$, and with $\mathbf{p} = (\underbrace{0, \dots, 0}_7, 1, 0)$.

Hence $T = Tr_H W \cup Tr_H^{\mathbf{p}}(Z) = T_1^0 \cup T_2^0 \cup \dots \cup T_7^0 \cup T_8^1 \cup R_{5,4}$ and $R = Res_H W \cup Res_H^{\mathbf{p}}(Z) = W \cup R_1^0 \cup R_2^0 \cup \dots \cup R_7^0 \cup R_8^1$.

We have that the ideal sheaf of $T_1^0 \cup T_2^0 \cup \dots \cup T_7^0 \cup R_{5,4}$ has $h^1 = 0$ and $h^0 = 2$ in degree 4, by using the previous case and the fact that $R_{5,4}$ is contained in a $(2, 3, 4)$ -point, so $h^0(\mathcal{I}_{T,H}(4)) = 0$ by Lemma 1.8, since T_8^1 is a 2-jet in $H \cong \mathbb{P}^4$. We also have $h^0(\mathcal{I}_R(3)) = 0$. In fact, let us denote by U the scheme $U = R_8^1 \cup W$. In order to apply Lemma 1.9 (the R_i^0 's are 2-jets) to get $h^0(\mathcal{I}_R(3)) = 0$, we need to show that $h^0(\mathcal{I}_{Res_H U}(2)) = 0$ and $h^1(\mathcal{I}_U(3)) = 0$. Since U is included in the union of four $(2, 3, 5)$ -points, which impose independent conditions in degree three (e.g. see [CGG1]), $h^1(\mathcal{I}_U(3)) = 0$ follows. Moreover, $Res_H(U)$ is made by three $(2, 3, 5)$ -points, and again $h^0(\mathcal{I}_{Res_H U}(2)) = 0$ is known by [CGG1].

Now, $h^0(\mathcal{I}_{T,H}(4)) = 0 = h^0(\mathcal{I}_R(3))$ imply $\mathbf{Prop}(5, 4)$ by Lemma 1.5, and we are done.

Case $n = 6$. Here we have $s_{6,4} = 16$, $r_{6,4} = 2$, while $s_{5,4} = 11$, $r_{5,4} = 5$. We want to use the Horace differential Lemma 1.5 with $Z = Z_1 \cup \dots \cup Z_{13} \cup R_{6,4}$, where Z_1, \dots, Z_{13} are $(2, 3, 6)$ schemes supported at generic points of H with direction $L_1, \dots, L_{12} \subset H$, while L_{13} is not in H , and we specialize $R_{6,4} \subset H$, as a generic 2-jet in H ; with $W = A_{14} \cup A_{15} \cup A_{16}$, and with $\mathbf{p} = (\underbrace{0, \dots, 0}_{11}, 1, 2, 0)$.

Hence $T = Tr_H W \cup Tr_H^{\mathbf{p}}(Z) = T_1^0 \cup T_2^0 \cup \dots \cup T_{11}^0 \cup T_{12}^1 \cup T_{13}^2 \cup R_{6,4}$ and $R = Res_H W \cup Res_H^{\mathbf{p}}(Z) = W \cup R_1^0 \cup R_2^0 \cup \dots \cup R_{11}^0 \cup R_{12}^1 \cup R_{13}^2$.

We have that $h^0(\mathcal{I}_{T,H}(4)) = 0$ by applying Lemma 1.1 and the previous case.

We also have $h^0(\mathcal{I}_R(3)) = 0$. In fact, let us denote by U the scheme $U = R_{12}^1 \cup R_{13}^2 \cup W$. In order to apply Lemma 1.9 (the R_i^0 's are 2-jets) to get $h^0(\mathcal{I}_R(3)) = 0$, we need to show that $h^0(\mathcal{I}_{Res_H U}(2)) = 0$ and $h^1(\mathcal{I}_U(3)) = 0$.

Since U is included in the union of five $(2, 3, 6)$ -points, which impose independent conditions in degree three (e.g. see [CGG1]), $h^1(\mathcal{I}_U(3)) = 0$ follows. Moreover, $Res_H(U)$ is made by three $(2, 3, 6)$ -points plus a 2-fat point inside $H \cong \mathbb{P}^5$. Since there is only one form of degree two passing through three generic $(2, 3, 6)$ -points in \mathbb{P}^6 , given by the hyperplane containing the three double lines, doubled, we get $h^0(\mathcal{I}_{Res_H U}(2)) = 0$.

Now, $h^0(\mathcal{I}_{T,H}(4)) = 0 = h^0(\mathcal{I}_R(3))$ imply **Prop**(6, 4) by Lemma 1.5, and we are done. \square

Now we come back to the proof of the Theorem for the remaining values of n, d ; we will work by induction on both n, d in order to prove statement **Prop**(n, d) for $n \geq 4, d \geq 5$ and for $n \geq 7, d = 4$. We divide the proof in 7 steps.

Step 1. The induction is as follows: we suppose that **Prop**(ν, δ) is known for all (ν, δ) such that $4 \leq \nu < n$ and $4 \leq \delta \leq d$ or $4 \leq \nu \leq n$ and $4 \leq \delta < d$ and we prove that **Prop**(n, d) holds.

The initial cases for the induction are given by Lemma 2.2, and we will also make use of the fact that **Reg**($n, 3$) with $n \geq 4$ and **Reg**($3, d$) with $d \geq 4$ hold respectively by assumption and by [B], while, by [CGG], we know everything about the Hilbert function of generic $(2, 3, n)$ -schemes when $d = 2$.

We will be done if we prove that **Degue**(n, d) and **Dime**(n, d) hold for $n \geq 4, d \geq 5$ and for $n \geq 7, d = 4$.

Step 2. Let us prove **Dime**(n, d). Notice that T is $\mathcal{O}_{\mathbb{P}^{n-1}}(d)$ -numerically settled in $H \cong \mathbb{P}^{n-1}$, hence **Dime**(n, d) is equivalent to $h^1(\mathcal{I}_{T,H}(d)) = 0$.

The scheme T is the generic union of $X_{s_{n-1,d}}$ with h 2-jets, of ϵ simple points and of δ 2-fat points, where $2h + \epsilon + n\delta = r_{n-1,d}$. Then **Dime**(n, d) holds for $n \geq 5$ and $d \geq 4$ since we are assuming that **Prop**($n-1, d$) is true and the union of h 2-jets, ϵ simple points and of δ 2-fat points can specialize to $R_{n-1,d}$ (see Lemma 1.10).

For $n = 4$ and $d \geq 5$, **Dime**($4, d$) holds, since we know that $h^1(\mathcal{I}_{X_{s_{3,d}}}(d)) = 0$ by [B] and in this case T is the generic union of $X_{s_{3,d}}$ with h 2-jets and ϵ simple points so we can apply Lemma 1.8.

Step 3. We are now going to prove **Degue**(n, d). Since the scheme R is the union of the scheme B and of $s_{n-1,d}$ 2-jets lying on H (see definitions of R and B above), we can use Lemma 1.9 ii). Hence, in order to prove that $\dim(I_R)_{d-1} = 0$, i.e. that **Degue**(n, d) holds, it is enough to prove that $(I_{Res_H(B)})_{d-2} = 0$ and that $\dim(I_B)_{d-1} \leq 2s_{n-1,d}$.

Step 4. Let us show that $(I_{Res_H(B)})_{d-2} = 0$. We set $t_{n,d} := s_{n,d} - s_{n-1,d} - h - \epsilon - \delta$. The scheme $Res_H(B)$ is given by W plus, if $\epsilon = 1$, one 2-fat point contained in H , plus, if $\delta = 1$, one simple point in H . W is the generic union of $R_{n,d}$ with $t_{n,d}$ $(2, 3, n)$ -points. Let I denote the ideal of these $t_{n,d}$ $(2, 3, n)$ -points; if we show that $I_{d-2} = 0$, then also $(I_{Res_H(B)})_{d-2} = 0$.

The idea is to prove that our $(2, 3, n)$ -points are “too many” to have $I_{d-2} \neq 0$ since they are more than $s_{n,d-2} + 1$; the only problem with this procedure is that there are cases (when $d-2 = 2$ or 3) where I_{d-2} may not have the expected dimension, so those cases have to be treated in advance.

First let $d = 4$ (and $n \geq 7$); if we show that $t_{n,4} > \frac{n}{2}$, then we are done, since $(I_{X_{s,n}})_2 = 0$ for $s > \frac{n}{2}$, by [CGG], Prop 3.3. The inequality $t_{n,4} > \frac{n}{2}$ is treated in Appendix A, A.2, and proved for $n \geq 7$, as required.

Now let $d = 5$ and $n = 4$; here we have that $s_{4,3} + 1 = 4$, but actually there is one cubic hypersurface through four $(2, 3, 4)$ -points in \mathbb{P}^4 ; nevertheless, since $t_{4,5} = 14 - 8 - 0 - 0 = 6$, and it is known (see [CGG] or [B]) that $(I_{X_{6,4}})_3 = 0$, we are done also in this case.

Eventually, for $d = 5$, $n \geq 5$, or in the general case $d \geq 6$, $n \geq 4$, if we show that $t_{n,d} \geq s_{n,d-2} + 1$, the problem reduces to the fact that $(I_{X_{s_{n,d-2}+1}})_{d-2} = 0$. If $d = 5$, we know that $(I_{X_{s_{n,3}+1}})_3 = 0$ by hypothesis, while for $d \geq 6$ we can suppose that $(I_{X_{s_{n,d-2}+1}})_{d-2} = 0$ by induction on d .

The inequality $t_{n,d} \geq s_{n,d-2} + 1$ is discussed in Appendix A, A.1, and proved for all the required values of n, d .

Thus the condition $(I_{Res_H(B)})_{d-2} = 0$ holds.

Step 5. Now we have to check that $\dim(I_B)_{d-1} \leq 2s_{n-1,d}$. Since $\deg Y_{n,d} = h^0(\mathcal{O}_{\mathbb{P}^n}(d))$ and $\deg T = h^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d))$, then $\deg R = h^0(\mathcal{O}_{\mathbb{P}^n}(d-1))$. The scheme R is the union of the scheme B and of $s_{n-1,d}$ 2-jets lying on H , so $\deg R = \deg B + 2s_{n-1,d}$. Hence $\dim(I_B)_{d-1} \leq 2s_{n-1,d}$ is equivalent to $h^1(\mathcal{I}_B(d-1)) = 0$ (and to $\dim(I_B)_{d-1} = 2s_{n-1,d}$).

Let us consider the case $n \geq 5$ first. Let Q be the scheme $Q = Z_{s_{n-1,d}+1} \cup \dots \cup Z_{s_{n,d}+h+\epsilon+\delta} \cup A_{s_{n,d}+h+\epsilon+\delta+1} \cup \dots \cup A_{s_{n,d}} \cup A_{s_{n,d}+1}$, where $A_{s_{n,d}+1}$ is a $(2, 3, n)$ scheme containing $R_{n,d}$. We have that B is contained in the scheme Q , which is composed by $s_{n,d} - s_{n-1,d} + 1$ generic $(2, 3, n)$ -points (notice that $2h + \epsilon + \delta \leq n + 1$, so $Z_{s_{n-1,d}+1}, \dots, Z_{s_{n,d}+h+\epsilon+\delta}$ are generic, since only the first h of the lines L_i are in H).

The generic union of $s_{n,d-1}$ generic $(2, 3, n)$ -points in \mathbb{P}^n is the scheme $X_{s_{n,d-1}}$; by induction, or by hypothesis if $d - 1 = 3$, we have $h^1(\mathcal{I}_{X_{s_{n,d-1}}}(d-1)) = 0$. Since $s_{n,d} - s_{n-1,d} + 1 \leq s_{n,d-1}$ (see *Step 6*), then $B \subset Q \subset X_{s_{n,d-1}}$ and we conclude by Remark 2.1 that $h^1(\mathcal{I}_B(d-1)) = 0$.

Step 6. We now prove the inequality: $s_{n,d} - s_{n-1,d} + 1 \leq s_{n,d-1}$ ($n \geq 5$).

We have $\deg Q = \deg B + 2h + \epsilon + n\delta + (2n + 1 - r_{n,d})$, in fact in order to “go from B to Q ”, we have to add a 2-jet to each of the R_i^1 (h in number), a simple point to $R_{s_{n-1,d}+h+\epsilon}^2$ if $\epsilon = 1$, a 2-fat point of H if $\delta = 1$ and something of degree $(2n + 1 - r_{n,d})$ to $R_{n,d}$.

Since $r_{n,d} \geq 0$ and $2h + \epsilon + n\delta = r_{n-1,d} \leq 2n - 2$, we have: $\deg Q = (2n + 1)(s_{n,d} - s_{n-1,d} + 2) \leq \deg(B) + 2n - 2 + 2n + 1 = \deg(B) + 4n - 1$.

Notice that $\deg(Y_{n,d-1}) = \deg(B) + 2s_{n-1,d}$, so we have: $(2n + 1)(s_{n,d} - s_{n-1,d} + 1) \leq \deg(Y_{n,d-1}) - 2s_{n-1,d} + 4n - 1$.

If we prove that $4n - 1 - 2s_{n-1,d} \leq 0$, we obtain: $(2n + 1)(s_{n,d} - s_{n-1,d} + 1) \leq \deg(Y_{n,d-1}) = (2n + 1)s_{n,d-1}$, and we are done.

The computations to get $4n - 1 - 2s_{n-1,d} \leq 0$ can be found in Appendix A.3.

Step 7. We are only left to prove that $h^1(\mathcal{I}_B(d-1)) = 0$ in case $n = 4$ ($d \geq 5$).

Recall that now $r_{3,d} = 2h + \epsilon \leq 6$, with $0 \leq h \leq 3$, $0 \leq \epsilon \leq 1$. If $r_{3,d} \leq 4$, we can apply the same procedure as in step 5, since the part of the scheme Q with support on H is generic in \mathbb{P}^4 . Hence we only have to deal with $r_{3,d} = 5, 6$.

The case $r_{3,d} = 5$ does not actually present itself; this can be checked by considering that

$$\binom{d+3}{3} = \frac{(d+3)(d+2)(d+1)}{6} = 7s_{3,d} + r_{3,d} \Rightarrow (d+3)(d+2)(d+1) = 42s_{3,d} + 6r_{3,d}.$$

Hence if $r_{3,d} = 5$, we get $42s_{3,d} + 30 = 7(6s_{3,d} + 4) + 2$, but it is easy to check that $(d+3)(d+2)(d+1)$ never gives a remainder of 2, modulo 7.

Thus we are only left with the case $r_{3,d} = 6$, when $h = 3$ and $\epsilon = 0$. In this case we have $d \equiv 3 \pmod{7}$, hence $d \geq 10$; it is also easy to check that $r_{3,d-1} = 3$ in this case.

We can add $2s_{3,d}$ generic simple points to B , in order to get a scheme B' which is $\mathcal{O}_{\mathbb{P}^4}(d-1)$ -numerically settled, so now $h^1(\mathcal{I}_B(d-1)) = 0$ is equivalent to $h^0(\mathcal{I}_{B'}(d-1)) = 0$ (by Remark 2.1).

We want to apply Horace differential Lemma again in order to prove $h^0(\mathcal{I}_{B'}(d-1)) = 0$; so we will define appropriate schemes Z_B , W_B and an integer vector \mathbf{q} , such that conditions a) and b) of Proposition 1.5 apply to them, yielding $h^0(\mathcal{I}_{B'}(d-1)) = 0$.

Consider the scheme $Z_B \subset \mathbb{P}^4$, given by $s_{3,d-1} - 1$ $(2, 3, 4)$ -schemes in \mathbb{P}^4 , such that their support is at generic points of H , and only for the last one of them the line L_i is not in H . Let $W_B \subset \mathbb{P}^4$ be given by $2s_{3,d}$ generic simple points, $s_{4,d} - s_{3,d} - s_{3,d-1} - 2$ generic $(2, 3, 4)$ -schemes, three generic $(2, 3, 3)$ -schemes in $H \cong \mathbb{P}^3$, and the scheme $R_{4,d}$. Let also $\mathbf{q} = (\underbrace{0, \dots, 0}_{s_{3,d-1}-3}, \underbrace{1}_1, \underbrace{2}_1)$.

Let $T_B = \text{Tr}_H(W_B) \cup \text{Tr}_H^{\mathbf{P}}(Z_B) = X_{s_{3,d-1}} \cup E \cup F$, and $R_B = \text{Res}_H(W_B) \cup \text{Res}_H^{\mathbf{q}}(Z_B)$.

We have that E and F are, respectively, a 2-jet and a simple point in H (they give the “remainder scheme” of degree 3, to get that T_B is $\mathcal{O}_{\mathbb{P}^3}(d-1)$ -numerically settled).

The scheme R_B is the union of $2s_{3,d}$ generic simple points, $s_{4,d} - s_{3,d} - s_{3,d-1} - 2$ generic $(2, 3, 4)$ -schemes, the scheme $R_{4,d}$, $s_{3,d-1}$ 2-jets in H , a $(2, 3, 3)$ -scheme in H and a 2-fat point of H doubled in a direction transversal to H .

If we show that $h^0(\mathcal{I}_{R_B}(d-2)) = 0 = h^0(\mathcal{I}_{T_B,H}(d-1))$, then we are done by Proposition 1.5.

We have $h^0(\mathcal{I}_{T_B,H}(d-1)) = 0$, since T_B is $\mathcal{O}_{\mathbb{P}^3}(d-1)$ -numerically settled, and is given by the union of $X_{s_{3,d-1}}$ (whose ideal sheaf has $h^1 = 0$ in degree $d-1$ by [B]) with a 2-jet and a simple point, so we can apply Lemma 1.8.

In order to show that $h^0(\mathcal{I}_{R_B}(d-2)) = 0$ we want to proceed as in Step 5, i.e by applying Lemma 1.9, since R_B , is made of $s_{3,d-1} - 3$ 2-jets union the $2s_{3,d}$ generic simple points and a scheme that we denote by R'_B . We will be done if we show that $h^0(\mathcal{I}_{\text{Res}_H(R_B)}(d-3)) = 0$ and $h^1(\mathcal{I}_{R'_B}(d-2)) = 0$.

The first condition will follow if $s_{4,d} - s_{3,d} - s_{3,d-1} - 2 \geq s_{4,d-3}$, the second condition (since R'_B is contained in the union of $s_{4,d} - s_{3,d} - s_{3,d-1} + 1$ generic $(2, 3, 4)$ -schemes) if $s_{4,d} - s_{3,d} - s_{3,d-1} + 1 \leq s_{4,d-2}$.

Both inequalities are proved in Appendix, A.4.

□

Thanks to some “brute force” computation by **COCOA**, we are able to prove:

Corollary 2.4 *For $4 \leq n \leq 9$, we have:*

i) $h^1(\mathcal{I}_{X_{s_{n,3}}}(3)) = 0$ and $h^0(\mathcal{I}_{X_{s_{n,3}+1}}(3)) = 0$, except for $n = 4$, in which case we have $h^0(\mathcal{I}_{X_{s,4}}(3)) = 0$ for $s \geq 5$.

ii) $h^0(\mathcal{I}_{Y_{n,d}}(d)) = h^1(\mathcal{I}_{Y_{n,d}}(d)) = 0$, for $d \geq 4$.

Proof: Part i) comes from direct computations using CoCoA ([CO]). Note that $s_{4,3} = 3$ and that $h^0(\mathcal{I}_{X_{4,4}}(3)) = h^1(\mathcal{I}_{X_{4,4}}(3)) = 1$, see [CGG1].

Part ii) comes by applying the Theorem and part i).

Coming back to the language of secant varieties, Theorem 2.2 and Corollary 2.4 give:

Corollary 2.5 *If Conjecture 1 is true for $d = 3$, then it is true for all $d \geq 4$. Moreover, for $n \leq 9$, Conjecture 1 holds.*

3. On Conjecture 2a. The case $n = 2$.

In this section we prove Conjecture 2a for $n = 2$.

We want to use the fact that $\sigma_s(O_{k,n,d})$ is defective if at a generic point its tangent space does not have the expected dimension; actually (see [BCGI]) this is equivalent to the fact that for generic $L_i \in R_1$, $F_i \in R_k$, $R = \kappa[x_0, \dots, x_n]$, $i = 1, \dots, s$ the vector space $\langle L_1^{d-k} R_k, L_1^{d-k-1} F_1 R_1, \dots, L_s^{d-k} R_k, L_s^{d-k-1} F_s R_1 \rangle$ does not have the expected dimension.

Via inverse systems this reduces to the study of $(I_Y)_d$, where $Y = Z_1 \cup \dots \cup Z_s$ is a certain 0-dimensional scheme in \mathbb{P}^n . Namely, the scheme Y is supported at s generic points $P_1, \dots, P_s \in \mathbb{P}^n$, at each of them $\deg(Z_i) = \binom{k+n}{n} + n$, and $I_{P_i}^{k+2} \subset I_{Z_i} \subset I_{P_i}^{k+1}$ (see Lemma 1.2).

When working in \mathbb{P}^2 , we can specialize the F_i 's to be of the form Π_i^k , where Π_i is a generic linear form through P_i . In this way we get a scheme $\bar{Y} = \bar{Z}_1 \cup \dots \cup \bar{Z}_s$, and the structure of each \bar{Z}_i is $((k+2)P_i \cap L_i^2) \cup (k+1)P_i$, where the line L_i is “orthogonal” to $\Pi_i = 0$, i.e. if we put $P_i = (1, 0, 0)$, $\Pi_i = x_1$ and $L_i = \{x_2 = 0\}$, the ideal is of the form: $((x_1, x_2)^{k+2} + (x_2)^2) \cap (x_1, x_2)^{k+1} = (x_1^{k+2}, x_1^{k+1}x_2, x_1^{k-1}x_2^2, \dots, x_2^{k+1})$.

Notice that the forms in $I_{\bar{Z}_i}$ have multiplicity at least $k+1$ at P_i and they meet L_i with multiplicity at least $k+2$; moreover the generic form in $I_{\bar{Z}_i}$ has L_i at least as a double component of its tangent cone at P_i .

When $F \in I_{\bar{Z}_i}$ and we speak of its “tangent cone” at P_i , we mean (with the choice of coordinates above) either the form in $\kappa[x_1, x_2]$ obtained by putting $x_0 = 1$ in F and considering the (homogeneous) part of minimum degree thus obtained, or also the scheme (in \mathbb{P}^2) defined by such a form.

When we will say that L_i is a “simple tangent” for F , we will mean that L_i is a reduced component for the tangent cone to F at P_i .

The strategy we adopt to prove Conjecture 2a is the following: if $(I_Y)_d$ does not have the expected dimension, i.e. $h^0(\mathcal{I}_Y(d))h^1(\mathcal{I}_Y(d)) \neq 0$, then the same happens for $\mathcal{I}_{\bar{Y}}(d)$; hence Conjecture 2a would be proved if we show that whenever $\dim(I_{\bar{Y}})_d$ is more than expected, then $h^1(\mathcal{I}_X(d)) > \max\{0, \deg(Y) - \binom{d+n}{n}\}$ or $h^0(\mathcal{I}_T(d)) > \max\{0, \binom{d+n}{n} - \deg(Y)\}$, where

$$X := (k+1)P_1 \cup \dots \cup (k+1)P_s \subset \mathbb{P}^2; \quad T := (k+2)P_1 \cup \dots \cup (k+2)P_s \subset \mathbb{P}^2.$$

The following easy technical Bertini-type lemma and its corollary will be of use in the sequel.

Lemma 3.1 *Let F, G be linearly independent polynomials in $\kappa[x]$. Then for almost any $a \in \kappa$, $F + aG$ has at least one simple root.*

Proof. Let M be the greatest common divisor of F and G with $F = MP$, $G = MQ$. Let us consider $PQ' - QP'$, where P' and Q' are the derivatives of P and Q , respectively. Since P and Q have no common roots, it easily follows that $PQ' - QP'$ cannot be identically zero.

For any $\beta \in \kappa$ which is neither a root for $PQ' - QP'$, nor for M , nor for Q , let

$$a = a(\beta) := -\frac{P(\beta)}{Q(\beta)},$$

so $(F + aG)(\beta) = M(\beta)(P + aQ)(\beta) = 0$, and $(F + aG)'(\beta) = (M'(P + aQ) + M(P' + aQ'))(\beta) = (M(P' + aQ'))(\beta) = (M(P' - \frac{P(\beta)}{Q(\beta)}Q'))(\beta) = (\frac{M}{Q})(\beta)(QP' - PQ')(\beta) \neq 0$, hence β is a simple root for $F + aG$. Since β assumes almost every value in κ , so does $a(\beta)$. \square

Corollary 3.2 *Let $P = (1, 0, 0) \in \mathbb{P}^2$. Let $f, g \in (I_P^{k+1})_d$, and $f, g \notin (I_P^{k+2})_d$. Assume that f, g , have different tangent cones at P . Then for almost any $a \in \kappa$, $f + ag$ has at least one simple tangent at P .*

Proof. The Corollary follows immediately from Lemma 3.1 by de-homogenising the tangent cones to f, g at P to get two non-zero and non-proportional polynomials $F, G \in \kappa[x]$. \square

It will be handy to introduce the following definitions.

Definition 3.3 Let $P \in \mathbb{P}^2$ and L be a line L through P . We say that a scheme supported at one point is of type Z' if its structure is $(k+1)P \cup ((k+2)P \cap L)$, and that it is of type \overline{Z} if its structure is $(k+1)P \cup ((k+2)P \cap L^2)$.

We will say that a union of schemes of types Z' and/or \overline{Z} is generic if the points of their support and the relative lines are generic.

The following lemma is the key to prove Conjecture 2a:

Lemma 3.4 *Let $\overline{Y} = \overline{Z}_1 \cup \dots \cup \overline{Z}_s \subset \mathbb{P}^2$ be a union of s generic schemes of type \overline{Z} , then either:*

(i) $(I_{\overline{Y}})_d = (I_T)_d$;

or

(ii) $\dim(I_{\overline{Y}})_d = \dim(I_X)_d - 2s$.

Proof. Notice that by the genericity of the points and of the lines, the Hilbert function of a scheme with support on P_1, \dots, P_s , formed by t schemes of type \overline{Z} , by t' schemes of type Z' and by $s - t - t'$ fat points of multiplicity $(k+1)$ depends only on s, t and t' .

Let W_t be a scheme formed by t schemes of type \overline{Z} and by $s - t$ fat points of multiplicity $(k+1)$. Let

$$\tau = \max\{t \in \mathbb{N} \mid \dim(I_{W_t})_d = \dim(I_X)_d - 2t\}.$$

If $\tau = s$, we have $W_s = \overline{Y}$ and $\dim(I_{W_s})_d = \dim(I_X)_d - 2s$, hence (ii) holds.

Let $\tau < s$: we will prove that $(I_{\overline{Y}})_d = (I_T)_d$. Let W be the scheme

$$W = W_\tau = \overline{Z}_1 \cup \dots \cup \overline{Z}_\tau \cup (k+1)P_{\tau+1} \cup \dots \cup (k+1)P_s.$$

and let

$$W'_{(j)} = \overline{Z}_1 \cup \dots \cup \overline{Z}_\tau \cup (k+1)P_{\tau+1} \cup \dots \cup Z'_j \cup \dots \cup (k+1)P_s, \quad \tau+1 \leq j \leq s,$$

$$W''_{(j)} = \overline{Z}_1 \cup \dots \cup \overline{Z}_\tau \cup (k+1)P_{\tau+1} \cup \dots \cup \overline{Z}_j \cup \dots \cup (k+1)P_s, \quad \tau+1 \leq j \leq s,$$

that is $W'_{(j)}$, respectively $W''_{(j)}$, is the scheme obtained from W by substituting the fat point $(k+1)P_j$ with a scheme of type Z' , respectively \overline{Z} , so

$$W \subset W'_{(j)} \subset W''_{(j)},$$

and $\deg W'_{(j)} = \deg W + 1$, $\deg W''_{(j)} = \deg W + 2$ (for $\tau = s - 1$, $W''_{(s)} = \overline{Y}$).

If $(I_{W''_{(j)}})_d = 0$, then trivially $(I_{\overline{Y}})_d = (I_T)_d = 0$ and we are done. So assume that $(I_{W''_{(j)}})_d \neq 0$.

By the definition of τ we have that $\dim(I_{W''_{(j)}})_d > \dim(I_X)_d - 2(\tau + 1) = \dim(I_W)_d - 2$, hence we get

$$0 \leq \dim(I_W)_d - \dim(I_{W''_{(j)}})_d \leq 1.$$

Let us consider the two possible cases.

Case 1: $\dim(I_W)_d - \dim(I_{W'_{(j)}})_d = 0$, $\tau + 1 \leq j \leq s$.

In this case we have $(I_W)_d = (I_{W'_{(j)}})_d$. This means that every form $F \in (I_W)_d$ meets the line L_j with multiplicity at least $k+2$; but since the line L_j is generic through P_j , this yields that every line through P_j is met with multiplicity at least $k+2$, hence

$$(I_W)_d \subset (I_{P_j}^{k+2})_d, \quad \text{for } \tau + 1 \leq j \leq s. \quad (1)$$

In particular, we have that

$$(I_W)_d = (I_{W''_{(s)}})_d. \quad (2)$$

Now consider the schemes

$$W_{(i,s)} = \overline{Z}_1 \cup \dots \cup \overline{Z}_{i-1} \cup (k+1)P_i \cup \overline{Z}_{i+1} \cup \dots \cup \overline{Z}_\tau \cup (k+1)P_{\tau+1} \cup \dots \cup (k+1)P_{s-1} \cup \overline{Z}_s, \quad 1 \leq i \leq \tau,$$

$$W'_{(i,s)} = \overline{Z}_1 \cup \dots \cup \overline{Z}_{i-1} \cup Z'_i \cup \overline{Z}_{i+1} \cup \dots \cup \overline{Z}_\tau \cup (k+1)P_{\tau+1} \cup \dots \cup (k+1)P_{s-1} \cup \overline{Z}_s, \quad 1 \leq i \leq \tau,$$

i.e. $W_{(i,s)}$ is the scheme obtained from W by substituting the fat point $(k+1)P_i$ to the scheme \overline{Z}_i and a scheme \overline{Z}_s , of type \overline{Z} , to the fat point $(k+1)P_s$, while $W'_{(i,s)}$ is the scheme obtained from $W_{(i,s)}$ by substituting a scheme Z'_i , of type Z' , to the fat point $(k+1)P_i$.

The schemes $W_{(i,s)}$ and W are made of τ schemes of type \overline{Z} and $s - \tau$ $(k+1)$ -fat points; the schemes $W'_{(i,s)}$ and $W'_{(s)}$ are made of τ schemes of type \overline{Z} , $s - \tau - 1$ $(k+1)$ -fat points and one scheme of type Z' . This yields that:

$$\dim(I_{W_{(i,s)}})_d = \dim(I_W)_d = \dim(I_{W'_{(s)}})_d = \dim(I_{W'_{(i,s)}})_d.$$

Hence every form $F \in (I_{W_{(i,s)}})_d$ meets the generic line L_i with multiplicity at least $k+2$, thus we get

$$(I_{W_{(i,s)}})_d \subset (I_{P_i}^{k+2})_d, \quad \text{for } 1 \leq i \leq \tau. \quad (3)$$

and from this and (2) we have

$$(I_{W_{(i,s)}})_d = (I_{W'_{(s)}})_d = (I_W)_d. \quad (4)$$

By (1), (3) and (4) it follows that $(I_W)_d = (I_T)_d$, hence, since $W \subset \overline{Y} \subset T$, we get (i).

Case 2: $\dim(I_W)_d - \dim(I_{W'_{(j)}})_d = 1$, $\tau + 1 \leq j \leq s$.

In this case we have

$$\dim(I_{W'_{(j)}})_d = \dim(I_{W''_{(j)}})_d.$$

Let $F \in (I_{W'_{(j)}})_d = (I_{W''_{(j)}})_d$; hence L_j appears with multiplicity two in the tangent cone of F . If $F \notin (I_{P_j}^{k+2})_d$, then let L'_j be a generic line not in the tangent cone of F at P_j . By substituting the line L'_j to L_j in the construction of $W'_{(j)}$, we get another form $G \in (I_W)_d$, $G \notin (I_{P_j}^{k+2})_d$, with the double line L'_j in its tangent cone. Then, by Corollary 3.2, the generic form $F + aG$ has a simple tangent at P_j , and this is a contradiction since a generic choice of the line L_j should yield $(I_{W'_{(j)}})_d = (I_{W''_{(j)}})_d$. Hence $F \in (I_{P_j}^{k+2})_d$, for $\tau + 1 \leq j \leq s$.

With an argument like the one we used in *Case 1*, we also get that $F \in (I_{P_j}^{k+2})_d$ for $1 \leq j \leq \tau$, and (i) easily follows. \square

Now we are ready to prove Conjecture 2a.

Theorem 3.5 *The secant variety $\sigma_s(O_{k,2,d})$ is defective if and only if one of the following holds:*

- (i) $h^1(\mathcal{I}_X(d)) > \max\{0, \deg(Y) - \binom{d+n}{n}\}$, or
- (ii) $h^0(\mathcal{I}_T(d)) > \max\{0, \binom{d+n}{n} - \deg(Y)\}$.

Proof. Since if Y is defective in degree d , then \bar{Y} is, but, by Lemma 3.4, either $\dim(I_{\bar{Y}})_d = \dim(I_X)_d - 2s$, hence

$$h^1(\mathcal{I}_X(d)) = h^1(\mathcal{I}_{\bar{Y}}(d)) - 2s > \max\{0, \deg(\bar{Y}) - \binom{d+n}{n}\} = \max\{0, \deg(Y) - \binom{d+n}{n}\},$$

or $(I_{\bar{Y}})_d = (I_T)_d$, hence

$$h^0(\mathcal{I}_T(d)) = h^0(\mathcal{I}_{\bar{Y}}(d)) > \max\{0, \binom{d+n}{n} - \deg(\bar{Y})\} = \max\{0, \binom{d+n}{n} - \deg(Y)\}.$$

\square

APPENDIX: Calculations

A.1 We want to prove that (for $n \geq 4$ and $d \geq 6$ or for $n \geq 5$ and $d = 5$):

$$s_{n,d} - s_{n-1,d} - h - \epsilon - \delta - 1 \geq s_{n,d-2}$$

Recall:

$$s_{n,d}(2n+1) + r_{n,d} = \binom{n+d}{d}; \quad s_{n-1,d}(2n-1) + r_{n-1,d} = \binom{n+d-1}{d}; \quad s_{n,d-2}(2n+1) + r_{n,d-2} = \binom{n+d-2}{d-2}.$$

Hence our inequality becomes:

$$\frac{1}{2n+1} \left[\binom{n+d}{d} - r_{n,d} \right] - \frac{1}{2n-1} \left[\binom{n+d-1}{d} - r_{n-1,d} \right] - h - \epsilon - \delta - 1 - \frac{1}{2n+1} \left[\binom{n+d-2}{d-2} - r_{n,d-2} \right] \geq 0$$

By using binomial equalities and reordering this is:

$$\frac{1}{2n+1} \left[\binom{n+d-1}{d} + \binom{n+d-2}{d-1} + \binom{n+d-2}{d-2} \right] - \frac{1}{2n-1} \binom{n+d-1}{d} + \frac{r_{n-1,d}}{2n-1} - h - \epsilon - \delta - 1 - \frac{1}{2n+1} \binom{n+d-2}{d-2} + \frac{1}{2n+1} (r_{n,d-2} - r_{n,d}) \geq 0$$

i.e.

$$\frac{1}{2n+1} \binom{n+d-2}{d-1} - \frac{2}{(2n+1)(2n-1)} \binom{n+d-1}{d} + \frac{r_{n-1,d}}{2n-1} - h - \epsilon - \delta - 1 + \frac{1}{2n+1} (r_{n,d-2} - r_{n,d}) \geq 0$$

By using binomial equalities again:

$$\frac{1}{2n+1} \binom{n+d-2}{d-1} - \frac{2}{(2n+1)(2n-1)} \left[\binom{n+d-2}{d} + \binom{n+d-2}{d-1} \right] + \frac{r_{n-1,d}}{2n-1} - h - \epsilon - \delta - 1 + \frac{1}{2n+1} (r_{n,d-2} - r_{n,d}) \geq 0$$

i.e.

$$\frac{1}{2n+1} \binom{n+d-2}{d-1} \left(1 - \frac{2}{2n-1}\right) - \frac{2}{(2n+1)(2n-1)} \binom{n+d-2}{d} + \frac{r_{n-1,d}}{2n-1} - h - \epsilon - \delta - 1 + \frac{1}{2n+1} (r_{n,d-2} - r_{n,d}) \geq 0$$

i.e.

$$\binom{n+d-2}{d-1} \frac{[2n(d-1)-3d+2]}{d(4n^2-1)} + \frac{r_{n-1,d}}{2n-1} - h - \epsilon - \delta - 1 + \frac{1}{2n+1} (r_{n,d-2} - r_{n,d}) \geq 0$$

Now, $\frac{r_{n-1,d}}{2n-1} \geq 0$, while $h + \epsilon + \delta \leq \frac{n}{2}$, and $r_{n,d-2} - r_{n,d} \geq -2n$, i.e. $\frac{1}{2n+1} (r_{n,d-2} - r_{n,d}) \geq -\frac{2n}{2n+1} \geq -1$, so our inequality holds if:

$$\binom{n+d-2}{d-1} \frac{[2n(d-1)-3d+2]}{d(4n^2-1)} - \frac{n}{2} - 2 \geq 0$$

It is quite immediate to check that the right hand side is an increasing function in d , e.g. by writing it as follows:

$$\binom{n+d-2}{n-1} [2n - 3 - \frac{2n+2}{d}] - (\frac{n}{2} + 2)(4n^2 - 1) \geq 0.$$

i.e.

$$\binom{n+d-2}{n-1} [2n - 3 - \frac{2n+2}{d}] - 2n^3 - 8n^2 + \frac{n}{2} + 2 \geq 0.$$

Let us consider the case $d = 6$ first; our inequality becomes:

$$\binom{n+4}{5} \frac{(10n-16)}{6} - 2n^3 - 8n^2 + \frac{n}{2} + 2 \geq 0.$$

i.e.

$$\frac{(n+4)(n+3)(n+2)(n+1)n(5n-8)}{360} - 2n^3 - 8n^2 + \frac{n}{2} + 2 \geq 0.$$

i.e.

$$\frac{(n+4)(n+3)(n+2)(n+1)n(5n-8)-20n^2(n+2)}{360} + \frac{n}{2} + 2 \geq 0.$$

i.e.

$$\frac{n(n+2)}{360} [(n+4)(n+3)(n+1)(5n-8) - 720n] + \frac{n}{2} + 2 \geq 0.$$

Which, for $n \geq 4$, is easily checked to be true. Hence we are done for $n \geq 4$, $d \geq 6$.

Now let us consider the case $d = 5$; our inequality becomes:

$$\binom{n+3}{4} \frac{(8n-13)}{5} - 2n^3 - 8n^2 + \frac{n}{2} + 2 \geq 0.$$

i.e.

$$\frac{(n+3)(n+2)(n+1)n(8n-13)}{120} - 2n^3 - 8n^2 + \frac{n}{2} + 2 \geq 0.$$

i.e.

$$(n^4 + 6n^3 + 11n^2 + 6n)(8n - 13) - 240n^3 - 960n^2 + 60n + 240 \geq 0.$$

i.e.

$$8n^5 + 35n^4 - 230n^3 - 1015n^2 - 18n + 240 \geq 0.$$

i.e.

$$n^3(8n^2 + 35n - 230 - \frac{1015}{n} - \frac{18}{n^2} + \frac{240}{n^3}) \geq 0.$$

Which, for $n \geq 6$, holds. So we are left to prove our inequality for $d = 5 = n$; in this case we have: $s_{5,5} = [\frac{272}{11}] = 24$, $s_{4,5} = [\frac{126}{9}] = 14$ and $r_{4,5} = 0$, hence $h = \epsilon = 0$, while $s_{5,3} = [\frac{56}{11}] = 5$; so: $s_{5,5} - s_{4,5} - 1 \geq s_{5,3}$ becomes: $24 - 14 - 1 \geq 5$, which holds.

□

A.2 We want to prove that, for all $n \geq 7$:

$$s_{n,4} - s_{n-1,4} - h - \epsilon - \delta > \frac{n}{2}$$

i.e.

$$\binom{n+4}{4}/(2n+1) - r_{n,4}/(2n+1) - \binom{n-1+4}{4}/(2n-1) + r_{n-1,4}/(2n-1) - h - \epsilon - \delta > \frac{n}{2}$$

i.e.

$$\frac{(n+4)(n+3)(n+2)(n+1)}{24(2n+1)} - \frac{(n+3)(n+2)(n+1)n}{24(2n-1)} - \frac{n}{2} - \frac{r_{n,4}}{(2n+1)} + \frac{r_{n-1,4}}{(2n-1)} - h - \epsilon - \delta > 0$$

Now:

$$\frac{r_{n,4}}{(2n+1)} \leq \frac{2n}{(2n+1)} < 1, \text{ hence } -\frac{r_{n,4}}{(2n+1)} > -1;$$

$$r_{n-1,4} \geq 0;$$

$$\text{and } h + \epsilon + \delta \leq \frac{n}{2}, \text{ i.e. } -h - \epsilon - \delta \geq -\frac{n}{2}.$$

Therefore we get:

$$\begin{aligned} & \frac{(n+3)(n+2)(n+1)}{24} \cdot \left[\frac{(n+4)}{(2n+1)} - \frac{n}{(2n-1)} \right] - \frac{n}{2} - \frac{r_{n,4}}{(2n+1)} + \frac{r_{n-1,4}}{(2n-1)} - h - \epsilon - \delta > \\ & \frac{(n+3)(n+2)(n+1)}{24} \cdot \left[\frac{n+4}{2n+1} - \frac{n}{2n-1} \right] - \frac{n}{2} - \frac{n}{2} - 1 = \\ & = \frac{(n+3)(n+2)(n+1)}{24} \cdot \frac{[(2n-1)(n+4) - n(2n+1)]}{(2n+1)(2n-1)} - n - 1 = \\ & (n+1) \left[\frac{(n+3)(n+2)(3n-2)}{12(4n^2-1)} - 1 \right] > 0 \end{aligned}$$

i.e.

$$(n+3)(n+2)(3n-2) - 12(4n^2-1) > 0$$

i.e.

$$3n^3 - 35n^2 + 8n > 0$$

which is true for $n \geq 12$.

Let us check the cases $n = 7, 8, 9, 10, 11$.

If $n = 7$ we have: $s_{7,4} = [\frac{1}{15}\binom{11}{4}] = 22$ (with $r_{7,4} = 0$); $s_{6,4} = 16$, since $\binom{10}{4} = 210 = 16 \cdot 13 + 2$, so $r_{6,4} = 2$ and $h = 1, \epsilon = \delta = 0$.

Our inequality becomes: $22 - 16 - 1 > 7/2$, which holds.

If $n = 8$ we have: $s_{8,4} = [\frac{1}{15}\binom{12}{4}] = 33$ (with $r_{8,4} = 0$); $s_{7,4} = 22$, $r_{7,4} = 0$ and $h = \epsilon = \delta = 0$.

Our inequality becomes: $33 - 22 > 4$, which holds.

If $n = 9$ we have: $s_{9,4} = [\frac{1}{15}\binom{13}{4}] = 47$ (with $r_{9,4} = 10$); $s_{8,4} = 33$, and $h = \epsilon = \delta = 0$.

Our inequality becomes: $47 - 33 > 9/2$, which holds.

If $n = 10$ we have: $s_{10,4} = [\frac{1}{15}\binom{14}{4}] = 66$ (with $r_{10,4} = 11$); $s_{9,4} = 47$, and $h = 5, \epsilon = \delta = 0$.

Our inequality becomes: $66 - 47 - 5 > 5$, which holds.

If $n = 11$ we have: $s_{11,4} = [\frac{1}{15}\binom{15}{4}] = 91$; $s_{10,4} = 66$, and $h = 5, \epsilon = 1, \delta = 0$.

Our inequality becomes: $91 - 66 - 5 - 1 > 11/2$, which holds.

□

A.3 We want to prove that, for $d \geq 5$, $n \geq 4$ or $d = 4$, $n \geq 7$:

$$4n - 1 \leq 2s_{n-1,d}. \quad (*)$$

Since $r_{n-1,d} \leq 2n - 2$, it is enough to prove that:

$$\frac{2}{2n-1} \left[\binom{n-1+d}{n-1} - 2n + 2 \right] \geq 4n - 1 \text{ which is:}$$

$$\binom{n-1+d}{n-1} \geq \frac{(4n-1)(2n-1)}{2} + 2n - 2 \text{ that is:}$$

$$\binom{n-1+d}{n-1} \geq 4n^2 - n - \frac{3}{2} \quad (**)$$

which is surely true if

$$\binom{n-1+d}{n-1} \geq 4n^2 - n \text{ is true.}$$

Notice that the function $\binom{n-1+d}{n-1}$ is an increasing function in d . For $d = 4$, the inequality becomes:

$$\frac{n(n^3+6n^2+11n+6)}{24} \geq 4n^2 - n, \text{ which can be written:}$$

$$n^3 + 6n^2 + 11n + 6 \geq 96n - 24, \text{ i.e.}$$

$$n^3 + 6n^2 - 85n + 30 \geq 0 \text{ which is surely true if the following is true:}$$

$$n^2 + 6n - 85 \geq 0. \text{ The last one is verified for } n \geq 8, \text{ so we are done for } d = 4 \text{ and } n \geq 8.$$

If $(n, d) = (7, 4)$, $s_{n-1,d} = 16$ since $\binom{10}{4} = 210 = 16 \cdot 13 + 2$, and $(*)$ becomes: $4 \cdot 7 - 1 \leq 2 \cdot 16$ which is true.

Since the function $\binom{n-1+d}{n-1}$ is an increasing function in d , we have proved the initial inequality for $d \geq 4$ and $n \geq 8$.

For $d = 5$ $(**)$ becomes: $n^5 + 10n^4 + 35n^3 - 430n^2 + 144n + 120 \geq 0$ which is true for $n = 5, 6, 7$. We have hence proved the initial inequality for $d \geq 5$ and $n \geq 5$.

If $(n, d) = (4, 5)$, $s_{n-1,d} = 8$ since $\binom{8}{3} = 8 \cdot 7$, and $(*)$ becomes: $4 \cdot 4 - 1 \leq 2 \cdot 8$ which is true.

For $d = 6$ $(**)$ becomes: $n(n+1)(n+2)(n+3)(n+4)(n+5) - 120(6)(4n^2 - n - 1) \geq 0$ which is true for $n = 4$. We conclude that the initial inequality is true for $d \geq 5$ and $n \geq 4$.

A.4 We want to show that (for $d \geq 10$): $s_{4,d} - s_{3,d} - s_{3,d-1} - 2 \geq s_{4,d-3}$ and $s_{4,d} - s_{3,d} - s_{3,d-1} + 1 \leq s_{4,d-2}$

The first inequality is equivalent to:

$$\left[\frac{1}{9} \binom{d+4}{4} \right] - \frac{1}{7} \binom{d+3}{3} + \frac{6}{7} - \frac{1}{7} \binom{d+2}{3} + \frac{3}{7} - 2 \geq \left[\frac{1}{9} \binom{d+1}{4} \right]$$

which follows if:

$$\frac{1}{9} \binom{d+4}{4} - \frac{1}{9} \binom{d+1}{4} \geq \frac{1}{7} \binom{d+3}{3} + \frac{1}{7} \binom{d+2}{3} - \frac{9}{7} + 4$$

i.e.

$$\frac{d+1}{9} \frac{[(d+4)(d+3)(d+2)-d(d-1)(d-2)]}{24} \geq \frac{1}{7} \left(\frac{(d+1)(d+2)(2d+3)}{6} \right) + \frac{19}{7}$$

i.e.

$$\frac{d+1}{9} \frac{(12d^2+24d+24)}{24} \geq \frac{1}{42}(d+1)(d+2)(2d+3) + \frac{19}{7}$$

i.e.

$$\frac{(d^2+2d+2)}{3} \geq \frac{2d^2+7d+6}{7} + \frac{114}{7(d+1)}$$

i.e.

$$d^2 - 7d - 4 \geq \frac{342}{d+1}$$

Which is easily checked to hold for $d \geq 10$.

Now let us consider the second inequality, which is equivalent to:

$$\left[\frac{1}{9} \binom{d+4}{4} \right] - \frac{1}{7} \binom{d+3}{3} + \frac{6}{7} - \frac{1}{7} \binom{d+2}{3} + \frac{3}{7} + 1 \leq \left[\frac{1}{9} \binom{d+2}{4} \right]$$

which follows if:

$$\frac{1}{9} \binom{d+4}{4} - \frac{1}{9} \binom{d+2}{4} \leq \frac{1}{7} \binom{d+3}{3} + \frac{1}{7} \binom{d+2}{3} - \frac{9}{7} - 3$$

i.e.

$$\frac{(d+1)(d+2)}{9} \frac{[(d+4)(d+3)-d(d-1)]}{24} \leq \frac{1}{7} \left(\frac{(d+1)(d+2)(2d+3)}{6} \right) - \frac{30}{7}$$

i.e.

$$\frac{(d+1)(d+2)}{9} \frac{(8d+12)}{24} \geq \frac{1}{42}(d+1)(d+2)(2d+3) - \frac{30}{7}$$

i.e.

$$\frac{1}{9} \geq \frac{1}{7} - \frac{180}{7(d+1)(d+2)(2d+3)}$$

Which is easily checked to hold for $d \geq 10$.

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